Single-shell return-to-the-origin probability diffusion MRI
measure under a non-stationary Rician distributed noise

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In a nutshell

**Problem:**

- The estimation of the Ensemble Average Propagator (EAP) and its related features such as the Return-To-the-Origin Probability (RTOP) measure requires a huge amount of densely sampled multi-shell \(q\)-space data.
- We analytically derive an alternative approach to retrieve the RTOP directly from a single-shell \(q\)-space data.

**Solution:**

- We provide a closed-form solution to correct noise-induced bias using a non-stationary log-Rician statistics.
- Under the narrow pulse assumption, the EAP in real space, \(P(R)|q\), is related to the diffusion signal attenuation \(E(q)\) in the \(q\)-space domain by means of the Fourier transform:
  \[
P(R)|q\rangle = \int_{R^3} E(q) \exp(-2\pi i q^T R) dq,
\]
- \(S(q)\) is the diffusion signal acquired at position \(q\),
- \(S_b\) is the baseline measured without a diffusion sensitization,
- \(q\) is the wave vector related to \(b = 4\pi^2\tau|q|^2\) with \(\tau\) being the effective diffusion time.

**Return-To-the-Origin Probability**

The probability in the origin indicates the EAP feature that the molecules minimally diffuse within the diffusion time and it is referred to as the RTOP measure:

\[
\text{RTOP} = \int_{R^3} E(q) dq.
\]

Considering a more general model beyond the diffusion tensor, i.e. \(E(q) = \exp(-bD(q))\), and assuming that the diffusion does not depend on the radial coordinate we can define the RTOP integral in a spherical system

\[
\text{RTOP} = \frac{\sqrt{\pi}}{4(4\pi)^{3/2}} \int_0^\infty \int_0^{\pi} \int_0^{2\pi} (D(\theta, \phi))^{-3/2} \sin\theta d\theta d\phi,
\]
- \(D(\theta, \phi)\) is the apparent diffusion coefficient.

**Numerical integration**

In order to numerically evaluate the RTOP integral, one can use a direct approach assuming that the element of the surface, \(\Delta S\), is inversely proportional to the number of gradients (i.e. \(\Delta S \propto 1/N_q\))

\[
\text{RTOP}\big|_{q}\big) = C \frac{1}{N_q} \sum_{j=1}^{N_q} \left( -\frac{1}{2} \log E(q) \right)^{-3/2} = C b^{3/2} \left( -\log E(q) \right)^{-3/2},
\]

- \(C = 8^{-1}(\pi\tau)^{-3/2}\) is a time-related constant.

Considering the second-order Taylor expansion of the expectation operator \(E\{f(X)\}\) given \(f(X) = X^{-3/2}\) we obtain the approximation

\[
E\big\{ X^{-3/2} \big\} \approx \frac{1}{8} \left( \frac{15}{32} \cdot \frac{2}{3} \cdot \delta + \frac{1}{2} \right),
\]

and finally redefine direct RTOP\(^3\) formulation using a sample mean estimator \(E\{X\} = (\log E(q))^p\)

**Non-stationary log-Rician bias**

Let us assume that the random variable \(\log S(q)\) follows a non-stationary log-Rician distribution with the underlying parameters \(A(x)\) and \(\sigma(x)\).

Assuming the random variables \(\log S(q)\) and \(\log S_b(q)\) are independent we state that

\[
\begin{align*}
E\left\{ \left( \log \frac{S(q)}{S_b(q)} \right)^2 \right\} &= E\left\{ (\log S(q))^2 \right\} - 2 E\{ \log S(q) \} E\{ \log S_b(q) \} \\
&+ E\{ \log S_b(q) \} E\{ \log S(q) \}
\end{align*}
\]

Given the asymptotic expansion of the expectation \(E\{ \log S(q) \}^2\) we revise the the RTOP\(^2\) formulation to handle the non-stationary log-Rician statistics

**New solution**

\[
\text{RTOP}\big|_{q}\big) = \frac{15}{8} C b^{3/2} \left( -\log E(q) \right)^{-3/2} - \frac{7}{8} C b^{3/2} \left( -\log E(q) \right)^{-3/2}.
\]

**Experimental results**

**Figure 1:** The RTOP estimation procedure.

**Figure 2:** (a) The RTOP measure obtained using the \(p/4\) samples (left) and the absolute error of the measures with reference to the fully-sampled data. (1) The genu of the corpus callosum (CC), (2) the anterior thalamic radiation and (3) the splenium of the CC. (b) Absolute components of the bias \(B(x)\) for \(N_q = 27\). (c) The mean relative error and the standard deviation of the RTOP measure.

**Table 1:** The correlation coefficient between the RTOP measures estimated under different maximal \(b\)-values (top) and under different techniques for same maximal \(b\)-value (bottom).