Gaussian related distributions

Santiago Aja-Fernández

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1 Gaussian related distributions

1. **Gaussian:** (Normal)

   PDF:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

   MGF:

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

   Main moments:

Mean	μ
Median	μ
Mode	μ
Variance	σ^2

2. **Rayleigh:**

$$R = \sqrt{X_1^2 + X_2^2} \qquad X_i \sim N(0, \sigma^2)$$

   PDF:

$$p(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{2}$$

   MGF:

$$M_X(t) = 1 + \sigma t e^{\sigma^2 t^2/2} \sqrt{\frac{\pi}{2}} \left(\text{erf}\left(\frac{\sigma t}{\sqrt{2}}\right) + 1\right)$$

   Raw moments:

$$\mu_k = \sigma^k 2^{k/2} \Gamma(1 + k/2)$$

   Main moments:

Mean	$\sigma\sqrt{\frac{\pi}{2}}$
Median	$\sigma\sqrt{\log 4}$
Mode	σ
Variance	$\frac{4-\pi}{2}\sigma^2$

$$\mu_1 = \sqrt{\frac{\pi}{2}}\sigma \qquad \mu_2 = 2\sigma^2$$

$$\mu_3 = 3\sqrt{\frac{\pi}{2}}\sigma^3 \qquad \mu_4 = 8\sigma^4$$

3. **Rician:**

$$R = \sqrt{X_1^2 + X_2^2} \quad X_i \sim N(A_i, \sigma^2)$$

$$R = |X| \quad X = N(A_1, \sigma^2) + jN(A_2, \sigma^2)$$

PDF:

$$p_M(M|A,\sigma) = \frac{M}{\sigma^2}\, e^{-\frac{M^2+A^2}{2\sigma^2}}\, I_0\left(\frac{AM}{\sigma^2}\right)\, u(M), \tag{3}$$

where $A = \sqrt{A_1^2 + A_2^2}$.

Raw moments:

$$\mu_k = \sigma^k 2^{k/2}\Gamma(1+k/2)L_{k/2}\left(-\frac{A^2}{2\sigma^2}\right)$$

where $L_n(x) = M(-n,1,x) = {}_1F_1(-n;1;x)$.

Main moments:

Mean	$\sigma\sqrt{\frac{\pi}{2}}L_{1/2}\left(-\frac{A^2}{2\sigma^2}\right)$
Median	
Mode	
Variance	$2\sigma^2 + A^2 - \frac{\pi\sigma^2}{2}L_{1/2}^2\left(-\frac{A^2}{2\sigma^2}\right)$

$$\mu_1 = \sqrt{\frac{\pi}{2}}L_{1/2}\left(-\frac{A^2}{2\sigma^2}\right)\sigma \qquad \mu_2 = A^2 + 2\sigma^2$$

$$\mu_3 = 3\sqrt{\frac{\pi}{2}}L_{3/2}\left(-\frac{A^2}{2\sigma^2}\right)\sigma^3 \qquad \mu_4 = A^4 + 8\sigma^2A^2 + 8\sigma^4$$

$$\mu_1 = \sqrt{\frac{\pi}{2}}L_{1/2}\left(-x\right)\sigma \qquad \mu_2 = 2\sigma^2(1+x)$$

$$\mu_3 = 3\sqrt{\frac{\pi}{2}}L_{3/2}\left(-x\right)\sigma^3 \qquad \mu_4 = 4\sigma^4(2+4x+x^2)$$

$$\begin{aligned} \text{Var} &= A^4 + 8\sigma^2A^2 + 8\sigma^4 \\ &\approx \sigma^2\left(1 - \frac{1}{4x} - \frac{1}{8x^2} + O(x^{-3})\right) \quad \text{with } x = \frac{A^2}{2\sigma^2} \end{aligned}$$

Series expansion of Hypergeometric Functions:

$$L_{1/2}(-x) = \frac{2\sqrt{x}}{\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}\sqrt{x}} + \frac{1}{16\sqrt{\pi}x^{3/2}} + O\left(x^{-5/2}\right)$$

$$L_{3/2}(-x) = \frac{4x^{3/2}}{3\sqrt{\pi}} + \frac{3\sqrt{x}}{\sqrt{\pi}} + \frac{3}{8\sqrt{\pi}\sqrt{x}} + \frac{1}{32\sqrt{\pi}x^{3/2}} + O\left(x^{-5/2}\right).$$

4. **Central Chi:** (Non-normalized version)

$$R = \sqrt{\sum_{i=1}^{2L} X_i^2} \quad X_i \sim N(0, \sigma^2)$$

$$R = \sqrt{\sum_{i=1}^{L} |Y_i|^2} \qquad Y_i = N(0,\sigma^2) + jN(0,\sigma^2)$$

PDF:

$$p(x|\sigma, L) = \frac{2^{1-L}}{\Gamma(L)} \frac{x^{2L-1}}{\sigma^{2L}} e^{-\frac{x^2}{2\sigma^2}} \tag{4}$$

Raw moments:

$$\mu_k = \sigma^k 2^{k/2} \frac{\Gamma(k/2+L)}{\Gamma(L)}$$

Main moments:

$$\mu_1 = \sqrt{2}\frac{\Gamma(L+1/2)}{\Gamma(L)}\sigma \qquad \mu_2 = 2L\sigma^2$$

$$\mu_3 = 3\sqrt{2}\frac{\Gamma(L+3/2)}{\Gamma(L)}\sigma^3 \qquad \mu_4 = 4L(L+1)\sigma^4$$

$$\text{Var} = 2\sigma^2 \left(L - \left(\frac{\Gamma(k/2+L)}{\Gamma(L)} \right)^2 \right)$$

5. **Non–central Chi:** (Non-normalized version)

$$R = \sqrt{\sum_{i=1}^{2L} X_i^2} \qquad X_i \sim N(A_i, \sigma^2)$$

$$R = \sqrt{\sum_{i=1}^{L} |Y_i|^2} \qquad Y_i = N(A_{1,i}, \sigma^2) + jN(A_{2,i}, \sigma^2)$$

PDF:

$$p(M_L|A_L, \sigma, L) = \frac{A_L^{1-L}}{\sigma^2} M_L^L e^{-\frac{M_L^2 + A_L^2}{2\sigma^2}} I_{L-1}\left(\frac{A_L M_L}{\sigma^2}\right) u(M_L), \tag{5}$$

with $A_L = \sqrt{\sum_i A_i^2}$

Main moments:

$$\mu_1 = \sqrt{2}\frac{\Gamma(L+1/2)}{\Gamma(L)} {}_1F_1\left(-\frac{1}{2}, L, -\frac{A_L^2}{2\sigma^2}\right)\sigma \qquad \mu_2 = A_L^2 + 2L\sigma^2$$

$$\mu_3 = 3\sqrt{2}\frac{\Gamma(L+3/2)}{\Gamma(L)} {}_1F_1\left(-\frac{3}{2}, L, -\frac{A_L^2}{2\sigma^2}\right)\sigma^3 \qquad \mu_4 = A_L^4 + 4(L+1)A_L^2\sigma^2 + 4L(L+1)\sigma^4$$

$$\mu_1 = \sqrt{2}\frac{\Gamma(L+1/2)}{\Gamma(L)} {}_1F_1\left(-\frac{1}{2}, L, -x\right)\sigma \qquad \mu_2 = 2\sigma^2(L+x)$$

$$\mu_3 = 3\sqrt{2}\frac{\Gamma(L+3/2)}{\Gamma(L)} {}_1F_1\left(-\frac{3}{2}, L, -x\right)\sigma^3 \qquad \mu_4 = 4\sigma^4(x^2 + 2(L+1)x + L(L+1))$$

$$\begin{aligned}\text{Var} &= 2\sigma^2 \left(1 + x - \left(\frac{\Gamma(L+1/2)}{\Gamma(L)} \right)^2 {}_1F_1{}^2\left(-\frac{1}{2}, L, -x\right) \right) \\ &\approx \sigma^2 \left(1 + \frac{1-2L}{4x} + \frac{3-8L+4L^2}{8x^2} + O(x^{-3}) \right) \qquad \text{with } x = \frac{A^2}{2\sigma^2}\end{aligned}$$

6. **Gamma:**

   PDF:
$$p_x(x) = x^{k-1}\frac{\exp(-x/\theta)}{\Gamma(k)\theta^k}u(x) \tag{6}$$

   MGF:
$$M_X(t) = (1-\theta t)^{-k} \text{ for } t < 1/\theta$$

   Main moments:

Mean	$k\theta$
Median	
Mode	$(k-1)\theta$
Variance	$k\theta^2$

$$k = \frac{\mu_1^2}{\text{Var}} \quad \theta = \frac{\text{Var}}{\mu_1} \quad \text{mode} = \mu_1 - \frac{\text{Var}}{\mu_1}$$

7. **K distribution:**

   PDF:
$$p_x(x) = \frac{2}{a\Gamma(N+1)}\left(\frac{x}{2a}\right)^{N+1} K_N\left(\frac{x}{a}\right)u(x) \tag{7}$$

   In ultrasound we consider the following parameters:
$$a = \frac{1}{b} \quad b = \sqrt{\tfrac{4\alpha}{E\{A^2\}}} = \sqrt{\tfrac{4\alpha}{2\sigma^2}} \quad N = \alpha - 1$$

   and then the PDF becomes:
$$p_x(x) = \frac{2b}{\Gamma(\alpha)}\left(\frac{xb}{2}\right)^{\alpha} K_{\alpha-1}(bx)\,u(x) \tag{8}$$

   General form of the PDF:
$$p_x(x) = \int_0^{\infty} p_R(x/y)p_\gamma(y)dy$$

   with $p_R(x/y)$ a Rayleigh distribution $R(\sigma^2 = 2a^2y)$:
$$p_R(x/y) = \frac{x}{2a^2y}e^{-\frac{x^2}{2(2a^2)y}}u(x)$$

   and $p_\gamma(y)$ a Gamma distribution $\gamma(N+1,1)$:
$$p_\gamma(y) = \frac{y^N}{\Gamma(N+1)}e^{-y}$$

   Moments:
$$E\{x^k\} = \frac{\Gamma(k/2+1)\Gamma(N+1+k/2)}{\Gamma(N+1)}(2a)^k$$

   Main moments (for ultrasound):
$$\mu_1 = \sqrt{2\sigma^2}\frac{\Gamma(3/2)\Gamma(\alpha+1/2)}{\sqrt{\alpha}\Gamma(\alpha)} \qquad \mu_2 = 2\sigma^2$$
$$\mu_3 = (2\sigma^2)^{3/2}\frac{\Gamma(5/2)\Gamma(\alpha+3/2)}{\alpha^{3/2}\Gamma(\alpha)} \qquad \mu_4 = 2(2\sigma^2)^2\left(1+\frac{1}{\alpha}\right)$$
$$\text{Var} = 2\sigma^2\left[1 - \frac{1}{\alpha}\left(\frac{\Gamma(3/2)\Gamma(\alpha+1/2)}{\Gamma(\alpha)}\right)^2\right]$$

2 Sample Local Variance (Gamma Approximation)

The (biased) SLV of an image $M(\mathbf{x})$ is defined as

$$\widehat{\text{Var}}(M(\mathbf{x})) = \frac{1}{|\eta(\mathbf{x})|}\sum_{\mathbf{p}\in\eta(\mathbf{x})} M^2(\mathbf{p}) - \left(\frac{1}{|\eta(\mathbf{x})|}\sum_{\mathbf{p}\in\eta(\mathbf{x})} M(\mathbf{p})\right)^2 \tag{9}$$

with $\eta(\mathbf{x})$ a neighborhood centered in $\mathbf{x}$. If $N = |\eta(\mathbf{x})|$, we define the random variable $V = \widehat{\text{Var}}(M(\mathbf{x}))$ with moments

$$\begin{aligned}
E\{V\} &= \left(1-\frac{1}{N}\right)(\mu_2-\mu_1^2)\\
E\{V^2\} &= \frac{1}{N^3}[(N^2-2N+1)\mu_4 + (N^3-3N^2+5N-3)\mu_2^2 + (-2N^3+12N^2-22N+12)\mu_2\mu_1^2\\
&\quad +(N^3-6N^2+11N-6)\mu_1^4 + (-4N^2+6N-4)\mu_3\mu_1]\\
\text{Var}(V) &= \frac{1}{N^3}\left[(-4N^2+8N-4)\mu_3\mu_1 + (8N^2-20N+12)\mu_2\mu_1^2 + (-N^2+4N-3)\mu_2^2\right.\\
&\quad \left.+(N^2-2N+1)\mu_4 + (-4N^2+10N-6)\mu_1^4\right]
\end{aligned}$$

For the main models:

1. **Rayleigh**

$$\begin{aligned}
E\{V\} &= 2\sigma_n^2\left(1-\frac{\pi}{4}\right)\left(\frac{N-1}{N}\right)\\
&\approx 2\sigma_n^2\left(1-\frac{\pi}{4}\right)\\
\sigma_V^2 &= \frac{\sigma_n^4}{N}\left(4+2\pi-\pi^2+O\left(\frac{1}{N}\right)\right)\\
&\approx \frac{\sigma_n^4}{N}(4+2\pi-\pi^2)
\end{aligned}$$

2. **Rice**

$$\begin{aligned}
E\{V\} &= \frac{(N-1)\sigma_n^2}{N}\left(1-\frac{1}{4x}-\frac{1}{8x^2}+O\left(x^{-3}\right)\right)\\
&\approx \frac{(N-1)\sigma_n^2}{N}\\
\sigma_V^2 &= \frac{(N-1)\sigma_n^4}{N^2}\left(2-\frac{1}{x}+\frac{3(2-5N+3N^2)}{8N(N-1)x^2}+O\left(x^{-3}\right)\right)\\
&\approx \frac{2(N-1)\sigma_n^4}{N^2}
\end{aligned}$$

for $x = \frac{A^2}{2\sigma_n^2}$.

3. **Central Chi**

$$\begin{aligned}
E\{V\} &= \frac{2\sigma_n^2(N-1)}{N}\left(L-\left(\frac{\Gamma(L+1/2)}{\Gamma(L)}\right)^2\right)\\
\sigma_V^2 &= \frac{4\sigma_n^4}{N}\left(L+8L\frac{\Gamma^2(L+1/2)}{\Gamma^2(L)}-4\frac{\Gamma^4(L+1/2)}{\Gamma^4(L)}-4\frac{\Gamma(L+1/2)\Gamma(L+3/2)}{\Gamma^2(L)}\right)+O\left(\frac{1}{N^2}\right)
\end{aligned}$$

with $K(L) = \left(L - \frac{\Gamma^2(L+1/2)}{\Gamma^2(L)}\right)$.

4. **Non–central Chi**

$$\begin{aligned} E\{V\} &= \frac{\sigma_n^2(N-1)}{N}\left(1+\frac{1-2L}{x}+\frac{3-8L+4L^2}{8x^2}+O\left(x^{-3}\right)\right) \\ &\approx \frac{\sigma_n^2(N-1)}{N} \\ \sigma_V^2 &= \frac{\sigma_n^4(N-1)}{N^2}\left(2+\frac{1-2L}{x}+O\left(x^{-2}\right)\right) \\ &\approx \frac{2\sigma_n^4(N-1)}{N^2} \end{aligned}$$

5. **K distribution**

$$\begin{aligned} E\{V\} &= 2\sigma^2\left(1-\frac{1}{N}\right)K_1(\alpha) \\ \sigma_V^2 &= \frac{\sigma^4}{N}(K_2(\alpha)+O(N^{-2})) \end{aligned}$$

with

$$K_1(\alpha) = 1-\frac{1}{\alpha}\left(\frac{\Gamma(3/2)\Gamma(\alpha+1/2)}{\Gamma(\alpha)}\right)^2$$

$$K_2 = 4+\frac{8}{a}+\frac{1}{a^2\Gamma(a)^4}\left[2\pi\Gamma(a+1/2)^2\Gamma(a)^2a-3\pi\Gamma(a+1/2)^2\Gamma(a)^2-\pi^2\Gamma(a+1/2)^4\right]$$

On the four cases, the PDF of the SLV may be approximated using a Gamma distribution. The mode of the distribution for each case is:

1. **Rayleigh**

$$\begin{aligned} \text{mode}\{V\} &= E\{V\}-\frac{\sigma_V^2}{E\{V\}} \\ &= \sigma_n^2\left(2-\frac{\pi}{2}-\frac{1}{N}\left(\frac{5\pi^2-16\pi}{2\pi-8}\right)+O(1/N^2)\right) \\ &\approx \sigma_n^2\left(2-\frac{\pi}{2}\right) \end{aligned}$$

2. **Rice**

$$\begin{aligned} \text{mode}\{V\} &= \sigma_n^2\left(1-\frac{3}{N}+O\left((\sigma/A)^2\right)\right) \\ &\approx \sigma^2 \end{aligned}$$

3. **Central Chi**

$$\begin{aligned} \text{mode}\{V\} &= 2\sigma_n^2\left(L-\frac{\Gamma^2(L+1/2)}{\Gamma^2(L)}\right)+O\left(\frac{1}{N}\right) \\ &\approx 2K(L)\sigma_n^2 \end{aligned}$$

4. **Non–central Chi**

$$\begin{aligned} \text{mode}\{V\} &= \sigma_n^2\left(1-\frac{3}{N}+O\left((\sigma_n/A)^2\right)\right) \\ &\approx \sigma_n^2 \end{aligned}$$

5. **K distribution**

$$\text{mode}\{V\} \quad \approx \quad \sigma_n^2 \left(\frac{4 - \frac{8K_1}{N} - \frac{K_2}{N} + \frac{4K_1}{N^2}}{2K_1(1-1/N)} \right)$$

3 Combination of Gaussian Variables

Let $X_i(A, \sigma)$, $i = 1, \cdots, N$ be a set of Gaussian random variables.

1. **Constant added and multiplied:**

$$S = aN(\mu, \sigma^2) + b \sim N(a\mu + b, a^2\sigma^2)$$

2. **Sum of zero mean Gaussian variables:** (IID)

$$S = N(0, \sigma_1^2) + N(0, \sigma_2^2) \sim N(0, \sigma_1^2 + \sigma_2^2)$$

$$S = \sum_{i=1}^{L} N_i(0, \sigma_i^2) \sim N(0, \sum_{i=1}^{L} \sigma_i^2)$$

3. **Sum of Gaussian variables:** (IID)

$$S = N(\mu_1, \sigma_1^2) + N(\mu_2, \sigma_2^2) \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$S = \sum_{i=1}^{L} N_i(\mu_i, \sigma_i^2) \sim N\left(\sum_{i=1}^{L} \mu_i, \sum_{i=1}^{L} \sigma_i^2 \right)$$

4. **Difference of Gaussian variables:** (IID)

$$S = N(\mu_1, \sigma_1^2) - N(\mu_2, \sigma_2^2) \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

5. **Product of Gaussian variables:** (IID)

$$S = N(\mu_1, \sigma_1^2) \times N(\mu_2, \sigma_2^2))$$

with PDF:

$$p(x) = \frac{1}{\pi\sigma_1\sigma_2} K_0 \left(\frac{|x|}{\sigma_1\sigma_2} \right)$$

6. **Sum of square zero mean Gaussian variables:** (IID)

$$S = \sum_{i=1}^{N} X_i^2(\sigma^2) \sim \gamma(N/2, 2\sigma^2) \sim \chi_N(\sigma^2/N)$$

with $X_i(\sigma^2) = N(0, \sigma^2)$ and $\chi_N(\sigma^2/N)$ a chi-square with N degrees of freedom.

7. **Square root of the sum of square Gaussian variables:** (IID)

$$S = \sqrt{X_1^2(0, \sigma^2) + X_2^2(0, \sigma^2)} \sim R(\sigma^2)$$

with $R(\sigma^2)$ a Rayleigh distribution.

$$S = \sqrt{X_1^2(A_1, \sigma^2) + X_2^2(A_2, \sigma^2)} \sim R_c\left(\sqrt{A_1^2 + A_2^2}; \sigma^2 \right)$$

with $R_c(A;\sigma^2)$ a Rician distribution.

$$S = \sqrt{\sum_{i_1}^{2L} X_i^2(0,\sigma^2)} \sim \chi_C(L,\sigma^2)$$

with $\chi_C(L,\sigma^2)$ a central Chi distirbution with parameters σ and L.

$$S = \sqrt{\sum_{i_1}^{2L} X_i^2(A_i,\sigma^2)} \sim \chi_N(A_L;L,\sigma^2)$$

with $\chi_N(A_L,L,\sigma^2)$ a non–central Chi distirbution with parameters σ, L and $A_L = \sqrt{\sum_i A_i^2}$.

8. **Sample variance of Gaussian**

$$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2 \sim \gamma\left(\frac{N-1}{2}, \frac{2\sigma^2}{N-1}\right)$$

4 Combination of Rayleigh Variables

Let $R_i(\sigma)$, $i = 1, \cdots, N$ be a set of Rayleigh random variables.

1. **Sum of square Gaussian variables:**

$$S = \sum_{i=1}^{N} X_i^2(\sigma^2) \sim \gamma(N/2, 2\sigma^2)$$

with $X_i(\sigma^2) = N(0,\sigma^2)$

2. **Sum of square Rayleigh variables:**

$$S = \sum_{i=1}^{N} R_i^2(\sigma^2) \sim \gamma(N, 2\sigma^2)$$

with PDF:

$$p_x(x) = x^{N-1}\frac{e^{-x/(2\sigma^2)}}{\Gamma(N)(2\sigma^2)^N}u(x)$$

3. **Sample mean of square Rayleigh variables:**

$$S = \frac{1}{N}\sum_{i=1}^{N} R_i^2(\sigma^2) \sim \gamma\left(N, \frac{2\sigma^2}{N}\right) \sim \chi_{2N}(\sigma^2/N)$$

with PDF:

$$p_x(x) = x^{N-1}\frac{N^N}{\Gamma(N)(2\sigma^2)^N}e^{-xN/(2\sigma^2)}u(x)$$

4. **Sum of Rayleigh variables:**

$$S = \sum_{i=1}^{N} R_i(\sigma^2)$$

with PDF (approx):

$$p_x(x) = x^{2N-1} \frac{e^{-x^2/(2bN)}}{2^{N-1} N^N b^N \Gamma(N)} u(x)$$

with

$$b = \frac{\sigma^2}{N} [(2N-1)!!]^{1/N}] \approx \sigma^2 \frac{2}{e} \approx \sigma^2 \frac{\pi}{4}$$

5. **Sample mean of Rayleigh variables:**

$$S = \frac{1}{N} \sum_{i=1}^{N} R_i(\sigma^2)$$

with PDF (approx):

$$p_x(x) = x^{2N-1} \frac{N^N}{2^{N-1} b^N \Gamma(N)} e^{-x^2 N/(2b)} u(x)$$

6. **Square sample mean of Rayleigh variables:**

$$S = \left(\frac{1}{N} \sum_{i=1}^{N} R_i(\sigma^2) \right)^2 \sim \chi_{2N}(b/N) \sim \gamma(N/2, 2b)$$

with PDF (approx):

$$p_x(x) = x^{N-1} \frac{N^N}{2^N b^N \Gamma(N)} e^{-xN/(2b)} u(x)$$

7. **Square root of sum of square Rayleigh variables:**

$$S = \sqrt{\sum_{i=1}^{N} R_i^2(\sigma^2)} \sim \chi(N, \sigma^2)$$

with $\chi(N, \sigma^2)$ a central chi with PDF:

$$p_x(x) = x^{2N-1} \frac{2}{\Gamma(N)(2\sigma^2)^N} e^{-x^2/(2\sigma^2)} u(x)$$

8. **Square root of sample mean of square Rayleigh variables:**

$$S = \sqrt{\frac{1}{N} \sum_{i=1}^{N} R_i^2(\sigma^2)} \sim \chi(N, \sigma^2/N)$$

with $\chi(N, \sigma^2/N)$ a central chi with PDF:

$$p_x(x) = x^{2N-1} \frac{2N^N}{\Gamma(N)(2\sigma^2)^N} e^{-x^2 N/(2\sigma^2)} u(x)$$

9. **Local sample variance of Rayleigh variables:**

$$V = \frac{1}{N} \sum_{i=1}^{N} R_i^2(\sigma^2) - \left(\frac{1}{N} \sum_{i=1}^{N} R_i(\sigma^2) \right)^2$$

For $x \geq 0$ and an even number of degrees of freedom the PDF is

$$p_V(x) = \frac{|x|^{N-1}\exp\left(-\frac{1}{4}\alpha^+ x\right)}{\Gamma(N)2\sigma_1^2\sigma_2^2(1-\rho^2)\gamma^-}\sum_{k=0}^{N-1}\frac{\Gamma(N+k)}{\Gamma(k+1)\Gamma(N-k)}\left(\frac{2}{\gamma^-|x|}\right)^k$$

with

$$\gamma^- = \frac{\left[(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2(1-\rho^2)\right]^{1/2}}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

$$\alpha^+ = \gamma^- + \frac{\sigma_2^2-\sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)},$$

$\sigma_1^2 = \frac{\sigma_n^2}{N}$, $\sigma_2^2 \approx \frac{\sigma_n^2}{N}\frac{\pi}{4}$ and ρ the correlation coefficient, which for large N may be approximated as

$$\rho \approx \frac{\pi}{4\sqrt{16\pi - 4\pi^2}} \approx 0.95$$

5 Combination of Rician Variables

Let $R_i(\sigma)$, $i = 1, \cdots, N$ be a set of Rician random variables.

1. **Sum of square Rician variables:**

$$S = \sum_{i=1}^{N} R_i^2(A_i, \sigma^2) \sim \chi^2_{2N}\left(\frac{x}{\sigma^2}, \frac{A_N^2}{\sigma^2}\right)$$

with $\chi^2_{2N}()$ a noncentral chi square with $A_N^2 = \sum_i A_i^2$, and PDF

$$p(x) = \frac{1}{2\sigma^2}\left(\frac{x}{A_N}\right)^{(N-1)/2} e^{-\frac{x+A_N^2}{2\sigma^2}} I_{N-1}\left(\frac{\sqrt{x}A_N}{\sigma^2}\right)$$

2. **Sample mean of square Rician variables:**

$$S = \frac{1}{N}\sum_{i=1}^{L} R_i^2(A_i, \sigma^2) \sim \chi^2_{2N}\left(\frac{xN}{\sigma^2}, \frac{A_N^2}{\sigma^2}\right)$$

with $\chi^2_{2N}()$ a noncentral chi square with PDF

$$p(x) = \frac{1}{2\sigma^2}\left(\frac{N}{A_N}\right)^{(N-1)/2} x^{(N-1)/2} e^{-\frac{xN+A_N^2}{2\sigma^2}} I_{N-1}\left(\frac{\sqrt{xN}A_N}{\sigma^2}\right)$$

3. **Sum of square Rician variables:** (approx.)

$$S = \sum_{i=1}^{L} R_i(A_i, \sigma^2)$$

with PDF

$$p(t) = \frac{t^L}{c_2^2}\left(\frac{c_1}{c_2 b}\right)^{L-1} e^{-\frac{t^2}{2c_2^2}-\frac{b^2}{2c_1^2}} I_{L-1}\left(\frac{tb}{c_1 c_2}\right)$$

with c_1 and c_2 constants, $t = x/\sqrt{L}$ and $b = \sqrt{\frac{LK\Omega}{K+1}}$, [**?**].

4. **Square of the sample mean of Rician variables:**

$$S = \left(\frac{1}{N}\sum_{i=1}^{L} R_i(A_i, \sigma^2)\right)^2 \sim \chi^2_{2N}\left(\frac{xN}{c_2^2}, \frac{b^2}{c_1^2}\right)$$

6 Combination of K Variables

Let $K_i(\sigma,\alpha)$, $i = 1, \cdots, N$ be a set of K-distributed random variables.

1. **Transformation of K-variables:**
   If
   $$S = f(K_i)$$
   then
   $$p_S(x) = \int_0^\infty p_T(x/y) p_\gamma(y) dy$$
   with $T = f(R_i)$ and R_i Rayleigh variables.

2. **Sum of square K variables:**
   $$S = \sum_{i=1}^{M} K_i^2(\sigma^2, \alpha)$$
   with PDF:
   $$p_S(x) = \frac{2}{\Gamma(M)\Gamma(\alpha)(2a)^{\alpha+M}} x^{\frac{M+\alpha}{2}-1} K_{\alpha-M}\left(\frac{\sqrt{x}}{a}\right) u(x)$$

3. **Sum of K variables:**
   $$S = \sum_{i=1}^{M} K_i(\sigma^2, \alpha)$$
   with PDF (approximation)
   $$p_S(x) = \frac{x^{M+\alpha-1}}{2^{\frac{M+\alpha}{2}-2}(2a^2 K3(M))^{\frac{M+\alpha}{2}}\Gamma(M)\Gamma(\alpha)} K_{\alpha-M}\left(\frac{x}{a\sqrt{K_3(M)}}\right) u(x)$$
   with
   $$K_3(M) = ((2M-1)!!)^{1/M}$$

7 Logarithmic transformation of RV

7.1 The logRayleigh distribution

Let $R(\sigma)$ be a Rayleigh RV with σ parameter, the transformation, la transformacin

$$lR(\sigma) = a\log(R(\sigma)) + b$$

is a log-Rayleigh withPDF

$$p_{lR}(x) = \frac{1}{a\sigma_a^2} e^{\frac{2x}{a}} e^{-\frac{e^{\frac{2x}{a}}}{2\sigma_a^2}}$$

with $\sigma_a^2 = \sigma^2 e^{\frac{2b}{a}}$.

Main moments

$$E\{x\} = \frac{a}{2}\left[\log(2\sigma_a^2) - \gamma\right]$$

where γ is the Euler's constant.

$$E\{x^2\} = \frac{a^2}{4}\left[\left(\log(2\sigma_a^2) + \frac{2b}{a} - \gamma\right)^2 + \frac{\pi^2}{6}\right]$$

$$\mathrm{Var}(x) = \frac{\pi^2 a^2}{24}$$

MGF:

$$\Phi_{lR}(\omega) = (2\sigma_a^2)^{j\omega\frac{a}{2}}\Gamma\left(j\frac{a}{2}\omega + 1\right)$$

7.2 Operation over log-Rayleigh

1. **Sum of logRayleigh**

$$S = \sum_{i=1}^{N} lR_i(\sigma)$$

   then the MGF is

$$\Phi_{\sum lR}(\omega) = (2\sigma_a^2)^{jN\omega\frac{a}{2}}\Gamma^N\left(j\frac{a}{2}\omega + 1\right)$$

   The mean is

$$E\{x\} = N\frac{a}{2}\left[\log(2\sigma_a^2) - \gamma\right]$$

2. **Sum of two logRayleigh**

$$S = lR_1(\sigma) + lR_2(\sigma)$$

   with PDF

$$p_2(t) = \frac{1}{a\sigma_a^4}e^{2t/a}K_0\left(\frac{e^{t/a}}{\sigma_a^2}\right)$$

7.3 The LogRician distribution

If we define the variable

$$Lr(\mathbf{x}) = \log\left(M(\mathbf{x})\right) = \log\left(\sqrt{(A + n_r(\sigma_n^2))^2 + n_i(\sigma_n^2)^2}\right) \tag{10}$$

this variable follows a LogRician distribution with PDF

$$p_L(x) = \frac{e^{2x}}{\sigma_n^2}e^{-\frac{e^{2x}+A^2}{2\sigma_n^2}}I_0\left(\frac{Ae^x}{\sigma_n^2}\right) \tag{11}$$

with $x \in (-\infty, +\infty)$. The moments of this distribution are

$$E\{x\} = \frac{1}{2}\Gamma^{(l)}\left(0, \frac{A^2}{2\sigma_n^2}\right) + \log(A) \tag{12}$$

$$E\{x^2\} = \frac{1}{4}\log^2\left(2\sigma_n^2\right) + \frac{1}{2}\log\left(2\sigma_n^2\right)\left[\Gamma^{(l)}\left(0, \frac{A^2}{2\sigma_n^2}\right) + \log\left(\frac{A^2}{2\sigma_n^2}\right)\right] + \frac{1}{4}\tilde{\mathrm{N}}_1\left(\frac{A^2}{2\sigma_n^2}\right) \tag{13}$$

$$\sigma_x^2 = \frac{1}{4}\left(\tilde{\mathrm{N}}_1\left(\frac{A^2}{2\sigma_n^2}\right) + \left[\Gamma^{(l)}\left(0, \frac{A^2}{2\sigma_n^2}\right) + \log\left(\frac{A^2}{2\sigma_n^2}\right)\right]^2\right) \tag{14}$$

$$\approx \frac{1}{4}\left(\tilde{\mathrm{N}}_1\left(\frac{A^2}{2\sigma_n^2}\right) - \log^2\left(\frac{A^2}{2\sigma_n^2}\right)\right) \tag{15}$$

with $\Gamma^{(l)}(a, b)$ the (lower) incomplete Gamma function and

$$\tilde{\mathrm{N}}_k(x) = e^{-x}\sum_{i=0}^{\infty}\frac{x^i}{\Gamma(i+1)}\left[\left(\psi(i+k)\right)^2 + \psi^{1)}(i+k)\right].$$

$\psi(x)$ is the polygamma function and $\psi^{1)}(x)$ is its first derivative.

It is easy to see that

$$\frac{1}{\sigma_x^2} \approx 2 \times \frac{A^2}{2\sigma_n^2}.$$

The same solution for the variance may be obtained using a Series expansion on eq. (10)

$$\begin{aligned} Lr(\mathbf{x}) &\approx \log(A) + \frac{n_1}{A} + \frac{n_2^2}{2A^2} - \frac{n_1^2}{2A^2} + ... \\ &\approx \log(A) + \frac{n_1}{A} \\ \sigma_{L_r}^2 &= \frac{\sigma^2}{A^2} \end{aligned}$$

7.4 The Log-Non Central Chi distribution

For parallel acquisitions, if we define the variable

$$Lc(\mathbf{x}) = \log\left(M_L(\mathbf{x})\right) = \frac{1}{2}\log\left(\sum_{l=1}^{L}\left((C_{l_r}(\mathbf{x}))^2 + (C_{l_i}(\mathbf{x}))^2\right)\right) \tag{16}$$

this variable follows a Log-Non central Chi distribution with PDF

$$p_{L_c}(x) = \frac{A_L^{1-L}}{\sigma_n^2} e^{x(N+1)} e^{-\frac{e^{2x}+A_L^2}{2\sigma_n^2}} I_{L-1}\left(\frac{A_L e^x}{\sigma_n^2}\right) \tag{17}$$

with $x \in (-\infty, +\infty)$. The moments of this distribution are

$$E\{x\} = \frac{1}{2}\log\left(2\sigma_n^2\right) + \frac{1}{2}\frac{A_L^2}{2L\sigma_n^2}\,{}_2F_2\left(1,1:2,1+L;-\frac{A_L^2}{2\sigma_n^2}\right) + \frac{1}{2}\psi(L) \tag{18}$$

$$\begin{aligned} E\{x^2\} &= \frac{1}{4}\left[\log^2\left(2\sigma_n^2\right) + 2\log\left(2\sigma_n^2\right)\left(\psi(L) + \frac{A_L^2}{2L\sigma_n^2}\,{}_2F_2\left(1,1:2,1+N;-\frac{A_L^2}{2\sigma_n^2}\right)\right)\right. \\ &\qquad \left. + \tilde{\mathrm{N}}_L\left(\frac{A_L^2}{2\sigma_n^2}\right)\right] \end{aligned} \tag{19}$$

$$\begin{aligned} \sigma_x^2 &= \frac{1}{4}\left[\tilde{\mathrm{N}}_L\left(\frac{A_L^2}{2\sigma_n^2}\right) - 2\log\left(2\sigma_n^2\right)\frac{A_L^2}{2L\sigma_n^2}\,{}_2F_2\left(1,1:2,1+N;-\frac{A_L^2}{2\sigma_n^2}\right)\right. \\ &\qquad \left. - \left(\psi(L) + \frac{A_L^2}{2L\sigma_n^2}\,{}_2F_2\left(1,1:2,1+N;-\frac{A_L^2}{2\sigma_n^2}\right)\right)^2\right] \end{aligned} \tag{20}$$

$$\approx \frac{1}{4}\left[\tilde{\mathrm{N}}_L\left(\frac{A_L^2}{2\sigma_n^2}\right) - \log^2\left(\frac{A_L^2}{2\sigma_n^2}\right)\right] \tag{21}$$

8 About the Coefficient of variation

8.1 CV of a Rayleigh distribution

The square coeffiecient of variation (CV) of a random variable x is defined to be

$$\mathcal{C}^2(x) = \frac{\mathrm{Var}(x)}{E^2\{x\}} = \frac{E\{x^2\} - E^2\{x\}}{E^2\{x\}} = \frac{E\{x^2\}}{E^2\{x\}} - 1 \tag{22}$$

If x is a random variable that follows a Rayleigh distribution, then

$$\mathcal{C}^2(x) = \frac{\frac{4-\pi}{2}\sigma_n^2}{\frac{2}{\pi}\sigma_n^2} = \frac{4-\pi}{\pi}$$

So, the CV of a Rayleigh distribution does not depend on the value of σ_n. In the paper, the value of the sample CV is used to tune the filter.

8.2 Sample CV of Rayleigh

Let $R_i(\sigma_n^2)$, $i = \{1, \cdots, N\}$ be a set of random variables with Rayleigh distribution. Then, if

$$X = \frac{1}{N} \sum_i R_i^2$$

$$Y = \left(\frac{1}{N} \sum_i R_i \right)^2$$

the sample CV may be defined as

$$Z = \frac{X - Y}{Y} = \frac{X}{Y} - 1 \tag{23}$$

Both X and Y are Chi-Square random variables.

$$X \sim \chi_{2N} \left(\frac{\sigma_n^2}{N} \right)$$

$$Y \sim \chi_{2N} \left(\frac{b}{N} \right)$$

The PDF of $U = \frac{X}{Z}$ assuming that X and Y are independent

$$f_U(u) = \frac{\Gamma(2N)}{\Gamma^2(N)} \left(\sigma_n^2 b \right)^N \frac{u^{N-1}}{(bu + \sigma_n^2)^{2N}} \tag{24}$$

and making $b \approx \sigma_n^2 \frac{\pi}{4}$

$$f_U(u) = \frac{\Gamma(2N)}{\Gamma^2(N)} \left(\frac{\pi}{4} \right)^N \frac{u^{N-1}}{(\pi u/4 + 1)^{2N}} \tag{25}$$

As $Z = U - 1$ then

$$f_Z(z) = \frac{\Gamma(2N)}{\Gamma^2(N)} \left(\frac{\pi}{4} \right)^N \frac{(z+1)^{N-1}}{(\pi(z+1)/4 + 1)^{2N}} \tag{26}$$

$$f_Z(z) \approx \frac{\Gamma(2N)}{\Gamma^2(N)} \left(\frac{4}{N} \right)^N \frac{(z+1)^{N-1}}{(z + 1 + \pi/4)^{2N}}$$

with mode

$$\text{mode}\{f_Z(z)\} = \frac{N-1}{N+1} \frac{4-\pi}{\pi} \approx \frac{4-\pi}{\pi}$$

8.3 CV in a Rician distribution

The square CV is

$$\begin{aligned}
\mathcal{C}(x)^2 &= \frac{E\{x^2\}}{E^2\{x\}} - 1 \\
&= \frac{2\sigma_n^2 + A^2}{\left(\sigma_n \sqrt{\frac{\pi}{2}} e^{-\frac{A^2}{4\sigma_n^2}} \left[\left(1 + \frac{A^2}{2\sigma_n^2} \right) I_0 \left(\frac{A^2}{4\sigma_n^2} \right) + \frac{A^2}{2\sigma_n^2} I_1 \left(\frac{A^2}{4\sigma_n^2} \right) \right] \right)^2} - 1 \\
&= \frac{2 + (A/\sigma_n)^2}{\frac{\pi}{2} e^{-\frac{A^2}{2\sigma_n^2}} \left[\left(1 + \frac{A^2}{2\sigma_n^2} \right) I_0 \left(\frac{A^2}{4\sigma_n^2} \right) + \frac{A^2}{2\sigma_n^2} I_1 \left(\frac{A^2}{4\sigma_n^2} \right) \right]^2} - 1
\end{aligned}$$

If $R = \frac{A^2}{2\sigma_n^2}$ we can write

$$\mathcal{C}^2 = \frac{2(1+R)}{\frac{\pi}{2} e^{-R} \left[(1+R)\, I_0 \left(\frac{R}{2} \right) + R I_1 \left(\frac{R}{2} \right) \right]^2} - 1$$