Variance Stabilization of Noncentral-Chi Data: Application to Noise Estimation in MRI

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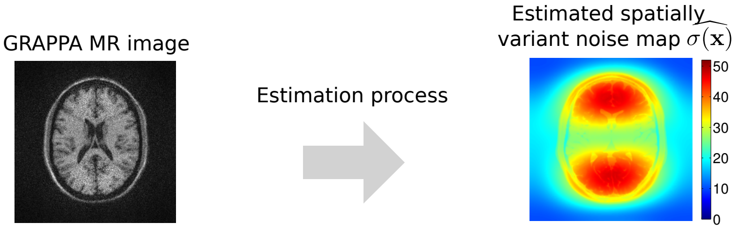
Outline

1. Problem statement
2. The variance-stabilizing transformation
3. Non-stationary nc-χ noise estimation
4. Numerical experiments
5. Final conclusions and remarks

Problem statement

Noise properties [Aja-Fernández, MRM 2011]:

- The noise is spatially variant.
- GRAPPA + SoS $\Longrightarrow$ nc-χ + *effective parameters* $L_{\text{eff}}(\mathbf{x})$ and $\sigma^2_{\text{eff}}(\mathbf{x})$



The state-of-the-art in noise estimation

Statistical models:

- nc-χ model – [Aja-Fernández, MRI 2014], [Tabelow, MedIA 2015],
- Gaussian model – [Goossens, ICIP 2006], [Pan, SPIE 2012], [Maggioni, SPIE 2012], [Aja-Fernández, ISBI 2015]
- Gaussian model + empirical corrections to nc-χ – [Veraart, MRM 2013], [Manjón, MedIA 2015], [Veraart, MRM 2016]

Drawbacks of the state-of-the-art:

- highly granular patterns,
- under-/overestimations for low SNR,
- computationally intensive schemes,
- reconstruction coefficients,
- multiple acquisitions.

The preliminary

Random variable $M_L \sim \text{nc-}\chi(A_T, \sigma_n, L) \implies \text{Var}\{M_L\}$ is signal-dependent!

We are looking for a function $f_{\text{stab}} : \mathrm{R} \to \mathrm{R}$:

- $\text{Var}\{f_{\text{stab}}(M_L)\}$ is signal-independent.
- $\text{Var}\{f_{\text{stab}}(M_L)\} = 1$

Solution #1

The first-order Taylor expansion of f_{stab} [Bartlett, Biometrics 1947].

$$f_{\text{stab}}(M_L|\sigma_n, L) = \int^{M_L} \frac{1}{\sqrt{\text{Var}\{M_L|\widetilde{A_T}, \sigma_n, L\}}} d\widetilde{A_T},$$

Problem! No closed-forms for $\mathrm{E}\{M_L\}$ and $\text{Var}\{M_L\}$

Asymptotic VST model

Random variable $M_L \sim$ nc-$\chi(A_T, \sigma_n, L)$

Solution #2 Use $M_L^2 \sim$ nc-$\chi^2(A_T, \sigma_n, L)$

$$\mathrm{E}\{M_L^2\} = A_T^2 + 2L\sigma_n^2, \quad \text{Var}\{M_L^2\} = 4A_T^2\sigma_n^2 + 4L\sigma_n^4$$

$$\text{Var}\{M_L^2|\mu_2, \sigma_n, L\} = 4\sigma_n^2\mu_2 - 4L\sigma_n^4 \implies \text{Cond. var. is in a closed-form!}$$

$$f_{\text{stab}}(M_L^2|\sigma_n, L) = \int^{M_L^2} \frac{1}{\sqrt{\text{Var}\{M_L^2|\widetilde{A_T}, \sigma_n, L\}}} d\widetilde{A_T} = \frac{1}{\sigma_n}\sqrt{M_L^2 - L\sigma_n^2}$$

Asymptotic model is not optimal for low SNRs!

Numerical VST model (for low SNRs)

A vector parameter $\boldsymbol{\Theta} = (\theta_1, \theta_2)$

$$f_{\text{stab}}(M_L^2|\sigma_n, L, \boldsymbol{\Theta}) = \frac{1}{\sigma_n}\sqrt{\max\{\theta_1^2 M_L^2 - \theta_2 L\sigma_n^2,\ 0\}}.$$

The cost function $J\colon \mathrm{R}^2 \mapsto \mathrm{R}$ to be minimized

$$\begin{aligned} J\left(f_{\text{stab}}(M_L^2|\sigma_n, L, \boldsymbol{\Theta})\right) &= \lambda_1 \cdot \varphi(1 - \mathrm{Var}\{f_{\text{stab}}(M_L^2|\sigma_n, L, \boldsymbol{\Theta})\}) \\ &+ \lambda_2 \cdot \varphi(\mathrm{Skewness}\{f_{\text{stab}}(M_L^2|\sigma_n, L, \boldsymbol{\Theta})\}) \\ &+ \lambda_3 \cdot \varphi(\mathrm{ExcessKurtosis}\{f_{\text{stab}}(M_L^2|\sigma_n, L, \boldsymbol{\Theta})\}). \end{aligned}$$

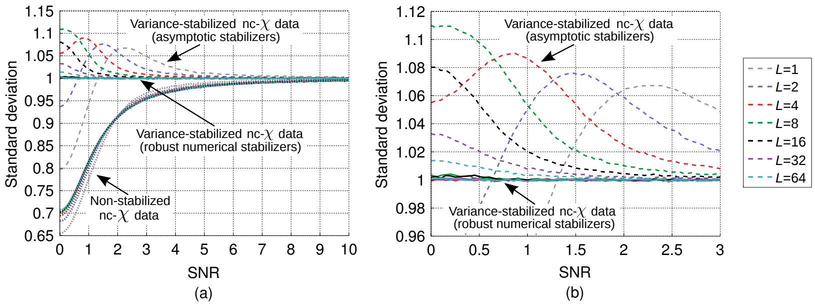
e.g., $\mathrm{Var}\left\{f_{\text{stab}}(M_L^2|\sigma_n, L, \boldsymbol{\Theta})\right\} = m_2 - m_1^2$

The r–th raw moment for f_{stab}–transformed nc-χ^2 RV

$$m_r = \mathrm{E}\{f_{\text{stab}}^r(M_L^2|\sigma_n, L, \boldsymbol{\Theta})\} = \int_0^\infty f_{\text{stab}}^r(\widetilde{M_L^2}|\sigma_n, L, \boldsymbol{\Theta})\, \underbrace{p(\widetilde{M_L^2}|A_T, \sigma_n, L)}_{\text{PDF of nc-}\chi^2\ \text{RV}}\, d\widetilde{M_L^2}$$

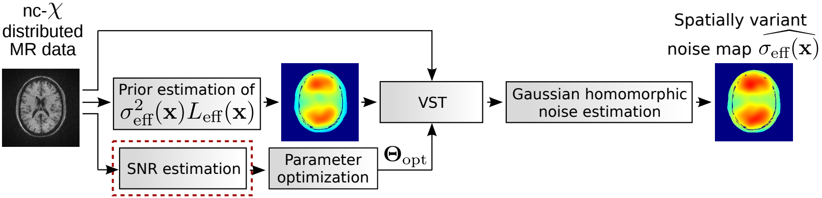
Evaluation of the proposed VST scheme

Standard deviation of the stabilized data



$$\text{SNR} = \frac{A_T}{\sqrt{L\sigma_n^2}}$$

General scheme for a non-stationary nc-χ noise estimation in GRAPPA MR.



$$\text{SNR}(\mathbf{x}) = \frac{A_T(\mathbf{x})}{\sqrt{\frac{L_{\text{eff}}(\mathbf{x})\sigma^2_{\text{eff}}(\mathbf{x})}{r}}},$$

- [Aja-Fernández, MRI 2013]
- [Tabelow, MedIA 2015]

Spatially variant noise estimation (1)

1. Stabilize the noisy MR image $I(\mathbf{x})$:

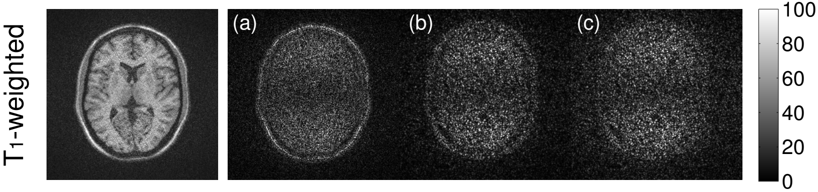
$$\widetilde{I}(\mathbf{x}) = \widehat{\sigma_{\text{eff}}(\mathbf{x})} \cdot f_{\text{stab}}(I^2(\mathbf{x})|\widehat{\sigma_{\text{eff}}(\mathbf{x})}, \widehat{L_{\text{eff}}(\mathbf{x})}, \boldsymbol{\Theta}_{\text{opt}}(\mathbf{x})).$$

2. The noise as AWGN component:

$$\widetilde{I}(\mathbf{x}) \approx A_T(\mathbf{x}) + N(\mathbf{x}; 0, \sigma^2_{\text{eff}}(\mathbf{x})) = A_T(\mathbf{x}) + \sigma_{\text{eff}}(\mathbf{x}) \cdot N(\mathbf{x}; 0, 1).$$

3. Center the data

$$\widetilde{I_{\text{C}}}(\mathbf{x}) = \widetilde{I}(\mathbf{x}) - \text{E}\{\widetilde{I}(\mathbf{x})\} = \sigma_{\text{eff}}(\mathbf{x}) \cdot N(\mathbf{x}; 0, 1),$$



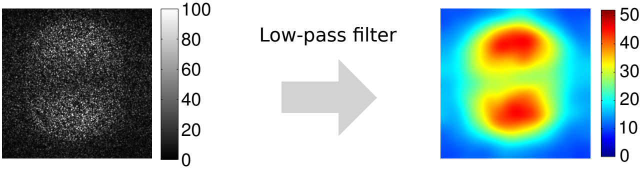
Spatially variant noise estimation (2)

5 Noise component representation:

$$\log|\widetilde{I_C}(\mathbf{x})| = \log|\sigma_{\text{eff}}(\mathbf{x}) \cdot N(\mathbf{x};0,1)| = \underbrace{\log \sigma_{\text{eff}}(\mathbf{x})}_{\text{low frequency}} + \underbrace{\log|N(\mathbf{x};0,1)|}_{\text{high frequency}}.$$

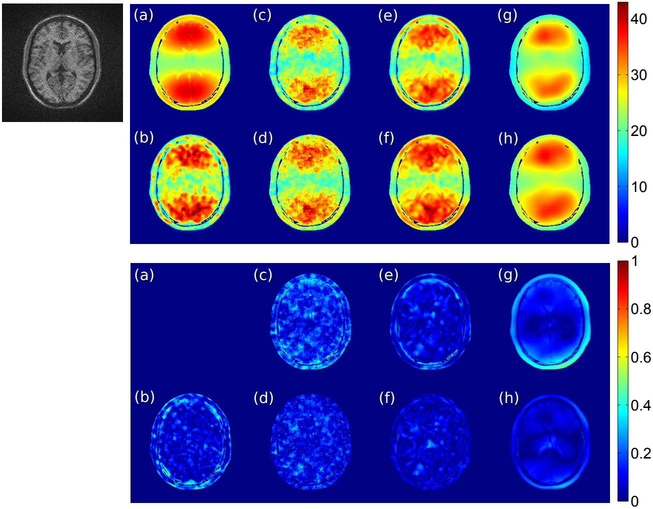
6 Gaussian homomorphic filter [Aja-Fernández, MedIA 2015]:

$$\widehat{\sigma_{\text{eff}}(\mathbf{x})} = \sqrt{2}\exp\left(\text{LPF}_{\sigma_f}\left\{\log\left|\widetilde{I_C}(\mathbf{x})\right|\right\} + \frac{\gamma}{2}\right).$$



Synthetic T_1-weighted GRAPPA MR ($L = 8$, $r = 2$, $\sigma_n = 15$, $\rho = 0.1$)

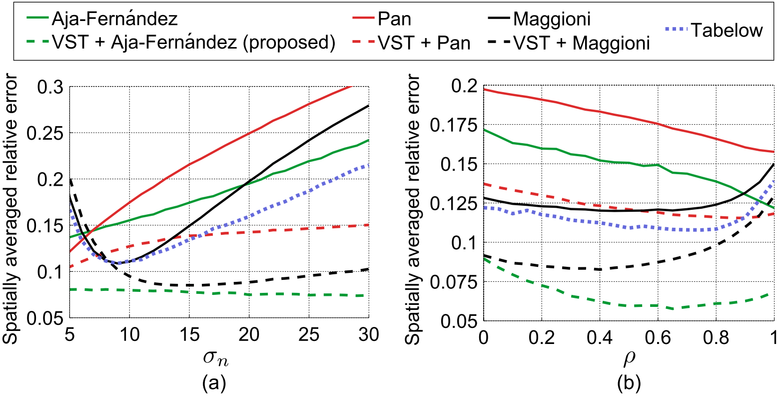
BrainWeb data



(a) Theoretical value; (b) Tabelow; (c) Pan; (d) VST + Pan, (e) Maggioni, (f) VST + Maggioni, (g) Aja-Fernández, (h) VST + Aja-Fernández (proposed).

Quantitative evaluation synthetic T_1 GRAPPA MR

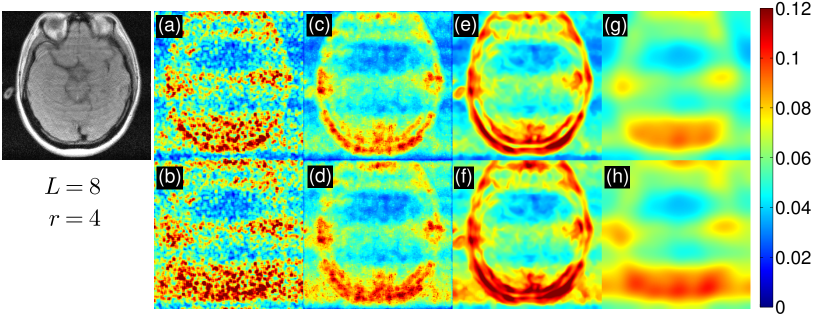
BrainWeb data



(a) $L = 8$, $r = 2$, $\rho = 0.1$

(b) $L = 8$, $r = 2$, $\sigma_n^2 = 150$

Real T_1-weighted GRAPPA MR
PULSAR data



$L = 8$

$r = 4$

(a) Goossens; (b) VST + Goossens; (c) Pan; (d) VST + Pan, (e) Maggioni, (f) VST + Maggioni, (g) Aja-Fernández, (h) VST + Aja-Fernández (proposed).

Final conclusions and remarks

The main advantages of the proposal:

1. It estimates the noise pattern for a one single GRAPPA MR image,
2. it is robust for the whole range of SNRs,
3. it does not require pre-scans or multiple acquisitions,
4. it does not need any technical details about the acqusition procedure,
5. it is not affected by a granular effect or a significant bias,
6. Any AWGN method can be employed in the VST framework.

Thank you for your attention!

AGH

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